Subtidal Frequency Fluctuations in Coastal Sea Level in the Mid and South Atlantic Bights: A Prognostic For Coastal Flooding

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ABSTRACT


An analytical model is used to determine spatial and temporal variations in coastal sea level. The model is developed for subtidal frequency motions in the viscous parameter regime, where advection of relative vorticity can be neglected with respect to production of relative vorticity by bottom Ekman layer pumping. The latter is then balanced by the topographically induced vertical velocity. The effects of the atmospheric wind stress on coastal sea level are assessed; and it is found that an upwelling (downwelling) favorable wind stress causes a continual drop (rise) in coastal sea level downstream from an initial cross-shelf location, with modifying effects of cross-shelf profile and initial location on the downstream variation in coastal sea level also being important. The model is applied over the Mid Atlantic Bight from Woods Hole, Massachusetts to Cape Hatteras, North Carolina and then to Charleston, South Carolina. Reasonable agreement exists between observations and model results which suggests that a predictive capability has been established; dimensionless mean squared errors range from 0.122 to 0.294 with a mean of 0.181 over the six test cases. The model can be driven by several days of forecast winds to determine the timing of coastal flooding, with linear superposition of location specific predicted astronomical tides onto the subtidal frequency model predictions.

ADDITIONAL INDEX WORDS: Sea-level change, coastal flooding, shelf dynamics, Ekman pumping.

INTRODUCTION

The response of continental shelf waters to the wind and outer boundary forcing is of considerable interest to those concerned with nearshore or estuarine processes or with setting coastal boundary conditions on numerical models. For example, it has been shown that wind induced variations dominate the subtidal frequency fluctuations of coastal sea level along the east coast of the U.S. (WANG, 1979; CHAO and PIETRAFESA, 1980), have a significant impact on the transport of particulate matter and fish larvae through coastal inlets (PIETRAFESA and JANOWITZ, 1988), or cause significant coastal flooding (NUMMENDAL et al., 1987). Thus, an understanding of the determinants of coastal sea level variation is of importance and will be considered here.

A steady state model is first developed and the results examined. The model is then extended to include time varying forcing and finally a comparison between observations and model predictions is undertaken, specifically for the Mid-Atlantic and the South Atlantic Bights as a test. It is anticipated that the results of this study will be of use to the U.S. National Weather Service in its attempts to predict coastal flooding events.

The literature on slope and shelf dynamics is extensive and growing. Here we discuss a few works relevant to the development of this paper. Several other papers are discussed further in the text. SCHWING et al. (1985) in their study of the dynamics of the South Carolina shelf found little phase lag between the wind and sea level in shallow water. They later found (1988) that for a period when the wind field was weak and disorganized, continental shelf waves may occur in the South Atlantic Bight. WANG (1982) and MIDDLTON (1987) showed that for wide shelves coastal sea level is relatively insulated from offshore pressure fields. BURRAGE et al. (1991) in a study of the central Great Barrier Reef showed that the alongshore flow is in geostrophic balance for pe-
riods exceeding 50 hours and that much of the motion of the shelf is wind driven. The elements of geostrophy, steadiness, wind-forcing and the neglect of offshore forcing all play a role in the present work.

A Steady State Model

Although we are ultimately concerned with the time varying response of coastal sea level to wind and outer boundary forcing, we first consider the steady state case. Consider a portion of the continental shelf bounded by a straight coastline with a depth that increases linearly with the offshore coordinate \( x \), i.e., \( h = h_e + \alpha x \). The coastal depth is \( h_e \) and \( \alpha \) is the diabathic bottom slope. The origin of the \( x \) axis is the coast, positive offshore, with the \( y \) coordinate taken in the alongshore direction ninety degrees to the left of the \( x \) axis, or towards higher latitude, see Figure 1. We consider the region \( 0 \leq x < \infty \) and \(-L \leq y \leq 0\). As we will show, the alongshore length \( L \) can be taken to be arbitrarily large. A uniform alongshore wind stress \( \tau_y \) is applied at the free surface and cross-shelf profiles of the sea level \( \eta(x, y) \) are specified at \( y = 0, -L \). Far offshore, \( \eta \) approaches zero. The density field is taken to be homogeneous and the Coriolis parameter \( f_o \) is assumed constant. Vertical mixing is parameterized by a constant eddy viscosity \( A_v \) and \( U \) and \( L_u \) are taken to be the offshore velocity and length scales respectively. The model is in the viscous domain, i.e., it is required that

\[
1 \gg E_0^c = (A_v/f_o h_e)^n \gg \epsilon = U/f_o L_u \tag{1}
\]

The surface and bottom Ekman layers are thus assumed thin compared to the total depth, and the motion is linear. We take \( u, v \), and \( w \) to be the components of the geostrophic velocity field to which corrections are added in the frictional boundary layers. Neglecting terms of relative order \( \epsilon \) and \( E_0^c \), the horizontal momentum balance becomes

\[
-f_u = g \frac{\partial \eta}{\partial y} \tag{2a}
\]

in the offshore direction, and

\[
f_v = g \frac{\partial \eta}{\partial x} \tag{2b}
\]

alongshore.

The continuity equation, with vanishing horizontal divergence from equation (2) through order \( E_0^c \) (\( >\epsilon \)) or equivalently the vorticity equation with advective terms neglected \( (O(\epsilon)) \), is

\[
\frac{\partial w}{\partial z} = 0. \tag{3}
\]

As the wind stress is uniform, the surface Ekman layer is non-divergent and we obtain for the geostrophic vertical velocity at the surface, \( z = 0 \),

\[
w(x, y, 0) = 0. \tag{4}
\]

By virtue of equations (3) and (4) the geostrophic vertical velocity throughout the column and at the bottom, \( z = -(h_e + \alpha x) \), must vanish. An expression for the vertical velocity at the bottom, given in PEDLOSKY (1987, p. 226, equation 4.9.36) after redimensionalizing is

\[
w(x, y, -h_e) = -\alpha u + \delta_x \zeta_x = 0 \tag{5}
\]

where \( \delta_x = (A_v/2f_o)^n \) and \( \zeta_x = \partial \eta/\partial x - \partial u/\partial y \). We note that \( \delta_x \) is one half the e-folding depth of a constant eddy viscosity bottom Ekman layer and a factor of \( 2\pi \) smaller than the conventional Ekman depth. Using Equations (2) and (5) we find that the equation governing sea level under the foregoing assumptions is as follows:

\[
\delta_x \left[ \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right] + \alpha \frac{\partial \eta}{\partial y} = 0. \tag{6}
\]

The first term in this equation represents Ekman pumping and the second the topographic effect.

The boundary conditions on this equation are:
at \( y = 0 \), \( \eta(x, 0) = \eta_i(x) \), \( \eta_{\infty} = \eta_i(0) \). 
\( \text{(7)} \)

at \( y = -L \), \( \eta(x, -L) = \eta_i(x) \),
\( \text{(8)} \)

and
\[ \text{as } x \to \infty, \quad \eta(x, y) \to 0. \]  
\( \text{(9)} \)

Note that \( \eta_{\infty} \) is simply the initial coastal sea level. At the coast, the net transport in the x direction is taken to vanish. This is composed of the surface Ekman transport \((\tau_b/\rho_f)\), the geostrophic transport \((u_\infty)\), and the bottom Ekman layer transport \((-v_b\delta_x)\). Using the geostrophic velocity from equation (2), we find
\[ \text{at } x = 0, \quad \tau_b \rho_f - h \frac{\partial \eta}{\partial x} - \delta_x \frac{\partial \eta}{\partial y} \frac{\partial \eta}{\partial x} = 0. \]  
\( \text{(10)} \)

In the Appendix, a constant eddy viscosity formulation (following WELANDER, 1957), of the governing equation and boundary conditions in water depths varying from small to large, compared to \( \delta_x \), is outlined. It is shown that, condition (10) is a valid approximation to utilizing a zero cross-shelf transport at vanishing depth; equation (6) is the deep water form of the general constant eddy viscosity governing equation; and the deep water form is valid in waters as shallow as \( h = 5 \delta_x \), where \( \delta_x \) is typically 2 m in deep water.

CSANADY (1978) is considered a similar problem but applied his coastal boundary condition at vanishing depth; in essence dropping the middle term in Equation (10). CSANADY reached this condition for this shallow depth by assuming that the alongshore wind stress balances the bottom alongshore stress as the depth vanishes. He then used the geostrophic balance in the cross shelf direction to relate the alongshore bottom stress to the cross shelf pressure gradient. However, if the geostrophic relation utilized by CSANADY is violated in shallow water, i.e., if the offshore pressure gradient balances the vertical gradient of offshore stress in these shallow depths rather than the Coriolis force, then this boundary condition is not valid. By applying our boundary condition in sufficiently deep water, this problem is avoided. Moreover, with differing boundary conditions our results differ significantly from those of CSANADY (1978). The Appendix provides a further discussion of these points.

Consider Equation 6. Note that the term \( \delta_x \eta \) is important only if the alongshore length scale is less than or equal to \( \delta_x/\alpha \) (~1 kilometer). Since we are considering alongshore length scales of order greater than 10 km, then we drop this term which decreases the order of the equation in y and eliminates the need for a downstream boundary condition (Equation 11). For solutions with a y length scale order \( \delta_x/\alpha \), and with larger offshore scales, Equation 6 reduces to \( \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial \eta}{\partial y} = 0. \) A boundary layer correction near \( y = -L \) of the form \( A(x) \eta \) \( e^{y/L} + L \delta_x \) can be added to the solution discussed below to satisfy any specified forcing at \( y = -L \). The function \( A(x) \) would equal the specified forcing \( \eta_i(x) \), less the solution obtained below, evaluated at \( y = -L \). The forcing near \( x = -L \) extends only a distance \( \delta_x \) upstream. Dropping the \( \frac{\partial^2 \eta}{\partial y^2} \) term, Equation 6 then becomes
\[ \frac{\partial^2 \eta}{\partial y^2} + \frac{\partial \eta}{\partial y} = 0. \]  
\( \text{(11)} \)

To put Equation 11 in more standard form, we redefine the alongshore variable and non-dimensionalize the variables as
\[ L_x = h_0/\alpha, \quad L_y = L_y h_0/\delta_x, \quad \eta_{\infty} = \eta / L_x / \rho g h_0, \quad \bar{x} = x/L_x, \quad \bar{y} = -y/L_y, \quad \bar{\eta} = \eta/\eta_{\infty}. \]  
\( \text{(12)} \)

This scaling results from giving equal weight to geostrophic and bottom layer cross-shelf transport in Equation 10 and to the balance required by Equation 11. The cross-shelf scale, \( L_x \), is the natural geometric scale over which the coastal depth doubles. In the model of CSANADY (1978), \( h_0 \) is set equal to zero and no natural scales occur. Next substitute the redefined variables in (12) into (7)-(11) and drop the overbars on dimensionless variables. Hereafter, dimensional quantities will be denoted by an asterisk. The following governing equations and boundary conditions now result:
\[ \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial \eta}{\partial y}, \]  
\( \text{(13a)} \)

at \( x = 0 \), \( \frac{\partial \eta}{\partial x} - \frac{\partial \eta}{\partial y} = T \),
\( \text{(13b)} \)

as \( x \to \infty \), \( \eta \to 0 \),
\( \text{(13c)} \)

at \( y = 0 \), \( \eta = \eta_i(x) \).  
\( \text{(13d)} \)

Equation 13a is the one-dimensional diffusion equation with y interpreted as the timelike variable. We note that in (13b) \( T = 1 \), but retain this symbol to track the wind stress response. This
problem, defined by 13a–13d, may be solved via Laplace Transform techniques. Let

\[ N(x, s) = \int_0^\infty e^{-sx}n(x, y) \, dy \]  

so

\[ \frac{\partial^2 N}{\partial x^2} - sN = -\eta_w(x), \]  

\[ \frac{\partial N}{\partial x}(0, s) - sN(0, s) = T/s - \eta_w, \]  

and

\[ N(x, s) \to 0, \quad \text{as } x \to \infty. \]

The solution to 13e–13h may be found via variation of parameters and then inverted to yield the following

\[ \eta(x, y) = -\mathcal{T} \left[ \frac{2\sqrt{\pi}}{y^{3/2}} e^{\frac{x^2}{4y}} - (1 + x) \text{Erfc}(\zeta) \right. \]

\[ + \int_0^\infty \eta_s(x') \, dx' \]

\[ \left. \times \left[ e^{\frac{x^2}{4y}} \text{Erfc}\left( \frac{x + x'}{2\sqrt{y}} \right) + \frac{1}{2\sqrt{\pi y}} \left( e^{-\frac{x^2}{4y}} - e^{-\frac{x^4}{4y}} \right) \right] \right) \]

(14)

where \( \zeta = x/2\sqrt{y}, \) and \( \text{Erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt. \)

As we can see from Equation 14, there are three distinct sources of sea level variation. The first two terms in this equation arise from the coastal boundary conditions (13b) and (13d) and more clearly in the transformation of this condition Equation 13g. Physically the first term results from the fact that the alongshore wind causes cross-shelf flow. The second term is associated with the production of coastal pressure gradients due to \( \eta_w. \) The pressure gradient driven flows caused by \( \eta_w \) must produce no cross-shelf transport at the coast. The third term in Equation 14, which we shall refer to as the profile term, arises from the fact that any vorticity present at the upstream boundary requires a cross-shelf motion, and hence an alongshore change in water level, to balance the bottom friction-induced vertical velocity. The distinct origin of the final two forcing functions \( \eta_s \) and \( \eta_w(x) \) can be seen as follows. Two different initial profiles with the same value of \( \eta_w \) will differ solely in the profile term, and two initial profiles which are identical except in the immediate vicinity of \( x = 0, \) where \( \eta_s \) is different, will differ solely in the \( \eta_w \) term in Equation 14.

Our interest in this paper lies in the source and manifestation of subinertial frequency variations in coastal sea level. Before we turn to this, let us consider the spatial location of the majority of sea level variation occurring in the cross-shelf direction. For large values of \( y, \) the second term in Equation 14 becomes \( \eta_s = \eta_w e^{i\tau/\sqrt{\pi y}}. \) The ratio \( \eta_s(x, y)/\eta_w(o, y) = e^{-i\tau}. \) When \( x = 1.04\sqrt{y} (\gg 1), \) this ratio is 0.75. Thus most of the sea level change observed at the coast has occurred in deep water. For large values of \( y, \) the first term in Equation 14 is

\[ \eta_s(x, y) = -\frac{2\sqrt{\pi}}{y^{3/2}} (e^{i\tau} - \sqrt{\pi} \text{Erfc}(\sqrt{\tau})). \]

The ratio of offshore to coastal sea level \( \eta_s(x, y)/\eta_w(x, y) \) is \( e^{i\tau/\sqrt{\pi y}}. \) When \( x = 0.5\sqrt{y} (\gg 1), \) this ratio is \(-0.63. \) Thus, again, most of the sea level change has occurred in deep water. This suggests that the choice of \( \delta \) should reflect deep water values. We now return to coastal sea level.

Coastal sea level is given by Equation 14 when evaluated at \( x = 0, \) i.e.,

\[ \eta(0, y) = -\mathcal{T} \left[ \frac{2\sqrt{\pi}}{y^{3/2}} - 1 + e^{i\tau} \text{Erfc}(\sqrt{\tau}) \right. \]

\[ + \int_0^\infty \eta_s(x') \, dx' \]

\[ \left. \times \text{Erfc}\left( \frac{x'}{2\sqrt{y}} + \sqrt{\tau} \right) \right) \]

(15)

If the alongshore stress varies with \( y, \) then \( T/s \) in (13g) becomes \( T(s) \) and (15) becomes

\[ \eta(0, y) = -\int_y^\infty T(y - y') e^{i\tau} \text{Erfc}(\sqrt{\tau}) \, dy' \]

\[ + \eta_s e^{i\tau} \text{Erfc}(\sqrt{\tau}) \]

\[ + \int_0^\infty \eta_s(x') \, dx' \]

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\[ \times e^{t + x} \text{Erfc} \left( \frac{x'}{2 \sqrt{y}} \right). \] (15a)

Equation 15a allows us to consider cases of large alongshore extent over which the alongshore wind may vary substantially.

DISCUSSION OF THE STEADY STATE RESULTS AND MODEL SENSITIVITY

In this section, we discuss the variation of coastal sea level with \( y \) for each of the forcing functions \( \tau_{\eta_\infty} \) and \( \eta_\sigma(\cdot) \). We first note that when \( y \ll 1 \),

\[ e^{t + x} \text{Erfc} \left( \sqrt{y} \right) = 1 - \frac{2}{\sqrt{\pi}} y^{1/2} + y + O(y^{3/2}), \]

and when \( y \gg 1 \),

\[ e^{t + x} \text{Erfc} \left( \sqrt{y} \right) = \frac{1}{\sqrt{\pi} y} + O(y^{-3/2}). \]

We first discuss the variation in \( \eta(0, y) \) due to the wind stress, i.e., the term multiplying \( T \) in Equation 15. For \( y \ll 1 \),

\[ \eta(0, y) \approx -y. \]

Thus, an upwelling favorable wind causes sea level to drop in the downstream direction. More generally, in the N hemisphere, sea level drops in the downwind direction. Also \( u^*(0, y) = \tau_{\eta_\infty} p_\sigma \delta_\sigma \), so for small \( y \), the offshore flux in the surface Ekman layer is balanced by geostrophic onshore flow. However, for \( y \gg 1 \),

\[ \eta(0, y) \approx \frac{2}{\sqrt{\pi}} y^{1/2}. \]

So, sea level continues to drop in the downstream direction although at less than the linear rate. Finally, as \( y \to \infty \),

\[ v^*(0, y) = \tau_{\eta_\infty} / p_\sigma \delta_\sigma, \]

so that the offshore surface layer flux is balanced by onshore bottom Ekman layer flow and \( u(0, y) \to 0 \). At \( y = 0.77 \) geostrophic onshore flow and bottom Ekman layer flow each take up half of the offshore surface layer flow. The term multiplying \(-T\) is plotted as the function \( B(y) \) in Figure 2.

We next consider the effect of initial coastal sea level, \( \eta_\infty \), on \( \eta(0, y) \). For \( y \ll 1 \),

\[ \eta(0, y) \approx \eta_\infty \left( 1 - \frac{2}{\sqrt{\pi}} y^{1/2} \right) \]

and hence initially this effect drops off rapidly. Next, for \( y \gg 1 \),

Figure 2. Alongshore variation of coastal sea level to initial coastal sea level \( (A(y)) \) and alongshore wind \( (- B(y)) \).

\[ \eta(0, y) \sim \eta_\infty \frac{1}{\sqrt{\pi} y} \]

and the effect drops off more slowly. The coefficient of \( \eta_\infty \) in Equation 15 is plotted as the function \( A(y) \) in Figure 2.

The effect of the initial cross-shelf profile on \( \eta(0, y) \) is somewhat more complex. If the initial profile is confined to the region \( 0 \leq x \leq x_m \) and if \( y \ll 1 \) and \( x_m \), then,

\[ \eta(0, y) \approx \frac{1}{\sqrt{\pi} y} \int_0^\infty \eta(x') dx'. \]

If \( x_m < 1 \) and if \( |\eta_\sigma(x)| \leq |\eta_\infty| \) then the coastal effect will dominate the profile effect. Now consider the case where \( \eta_\sigma(x) = \delta(x-x_m) \), i.e., a concentrated offshore spike in the initial cross-shelf profile. Then

\[ \eta(0, y) = e^{t + x} \text{Erfc} \left( \frac{x_m}{2 \sqrt{y}} \right). \] (16)

Since \( \eta(0, y) \) vanishes as \( y \to 0 \) and \( y \to \infty \), the effect of this spike on coastal sea level will reach a maximum, \( \eta_m \), at some value \( y \), say \( y_m \). Differentiating Equation 16 with respect to \( y \) and setting the result equal to zero allows us to solve for \( y_m \) as a function of \( x_m \). We find that for \( x_m \ll 1 \),

\[ y_m = x_m / 2 \]

and

\[ \eta_m = 1 - \sqrt{2 x_m / \pi}, \] (17)

while for \( x_m \gg 1 \),

\[ y_m = -x_m / 2 \]
response is proportional to $h_e$ and $(\alpha \delta_a)^{-\nu}$. For small $y$, this response is proportional to $1 - 2/\sqrt{\pi y}$. We use $A(y)$ in Figure 2 to evaluate this response. If we start with $y = 1$, and double $(\alpha \delta_a)$ to increase $y$ to 2, the response is decreased by a factor of 0.81. If we double $h_e$, $y$ is decreased from 1 to 0.25, and the response is increased by a factor of 1.54.

The uncertainty of $\alpha \delta_a$, due to the uncertainty in $\delta_a$, is larger than that in $h_e$. The percentage change in the response is less than or equal to one half the percentage change in $\alpha \delta_a$. Thus the model is not very sensitive to the most poorly known parameter.

We now turn to the effects of time varying forcing functions with an inertial cutoff.

A MODEL FOR SUBINERTIAL SEA-LEVEL VARIATIONS

We now consider motions with time scales in excess of $f^{-1}$. Under the assumptions of Equation 1, vorticity Equation 3 is replaced with

$$\frac{\partial \delta_a*}{\partial t^*} = \frac{f}{\delta_a*} \frac{\partial w^*}{\partial x^*}. \quad (19)$$

Using Equation 5, then (19) becomes

$$\frac{\partial \delta_a*}{\partial t^*} = \frac{f}{h^*(x)} \left( \delta_a* - \alpha u^* \right). \quad (20)$$

We now consider the case when the forcing functions $(\tau^*, \eta_{in}(x))$ are slowly varying functions of time. The effects of $\tau^*(t)$ and $\eta_{in}(t)$ are applied at the coast where the coastal spin up time $h_e/\delta_a$ is typically about one half a day. If the time scales of these coastal forcing functions are considerably larger than one day we might expect coastal sea level to respond in a quasi-steady manner to these forcing functions. The conclusion was also reached by WRIGHT (1986). The response to the initial profile in general would be quite complex as the spin up time over the outer shelf can be several days to a week. If the cross shelf profile at $y = 0$ is confined to the region $0 \leq x \leq x_0 < 1$, we can neglect the profile term and take

$$\eta^*(0, y, t^*) = \eta_{in}^*(t^*)A(y) - \frac{\tau_{sys}}{\tau_{sys}} \eta_{in}B(y) \quad (21a)$$

where, as previously denoted,

$$A(y) = e^v \text{Erfc}(\sqrt{y}) \quad (21b)$$

$$B(y) = \frac{2}{\sqrt{y}} - 1 + e^v \text{Erfc}(\sqrt{y}) \quad (21c)$$
We now apply Equation 21 to the Middle Atlantic Bight and then to the South Atlantic Bight (both shown in Figure 3), by driving the model with actual wind data and then make a comparison of model results to observed coastal sea level, to assess the applicability of the model in these regions.

**COMPARISON WITH OBSERVATIONS**

In this section we utilize Equation 21 to predict alongshore sea level variations in two sections of the Mid-Atlantic Bight and one section of the South Atlantic Bight for an independent evaluation of the technique. In utilizing this equation, we neglect the contribution of the profile term. We do this not because the term is not significant, as it may well be for small \( \gamma \), but rather because initial profile data are not available.

We first compare predictions of Equation 21 with observations in the Mid-Atlantic Bight. Forty hour low passed data is utilized in all comparisons. First, consider a region of shelf in the Mid-Atlantic Bight ranging from Woods Hole, Massachusetts, to Sandy Hook, New Jersey, see Figure 3. Sea level at Woods Hole and the alongshore wind at the Brookhaven National Laboratory tower are taken as the forcing functions and sea level at Montauk, (Long Island) New York, and Sandy Hook are computed and compared with observations. We take \( h_{0} = 10 \text{ m} \), \( \theta_{0} = 2 \text{ m} \), and \( \alpha = 0.62 \times 10^{-3} \); we then find that \( L_{x} \) is 80 km (\( = h_{0}/\alpha \)). We note, while our assumed value of \( \theta_{0} \) appears small, the equivalent value of \( \theta_{0} \) utilized by Hickey and Pola (1983) in their study of west coast sea level changes is even smaller, i.e., approximately 1 meter. Their linear bottom coefficient \( \lambda (= 0.01 \text{ cm/sec}) \) is equivalent to a value of \( \delta_{0} \) equal to \( \lambda/\gamma \). This of course does not justify the use of \( \delta_{0} = 2 \text{ m} \). As discussed earlier, \( \delta_{0} \) should assume a deep water value where the bottom turbulence is driven by the overlying geostrophic currents. Following Csanady (1982), we take \( A_{v} = u_{0}^{2}/20f \) which yields \( \delta_{0} = u_{0}^{2}/20f \), where \( u_{0}^{2} \) is the bottom frictional velocity. If we now take \( u_{0}^{2} = C_{0}q_{0}^{2} \) where \( q_{0} \) is the overlying geostrophic current, then taking \( C_{0} = 1.6 \times 10^{-3} \) and \( q_{0} = 10 \text{ cm/sec} \) we find \( \delta_{0} = 2 \text{ m} \). Rational arguments can be made for using larger values of \( C_{0} \) and/or smaller values of \( q_{0} \) so that \( 2 \text{ m} \) appears a reasonable choice. The sensitivity to the choice of \( \delta_{0} \) has been discussed previously and is not large. Comparisons are given in Figure 4. There is an increase in amplitude of both observed and predicted sea level in going down the coast and agreement is quite good. Because of the relatively small value of \( L_{x} \) in this region, the response to the \( n_{x} \) forcing drops rapidly in the alongshore directions. The values for \( A(y) \) drop to 0.40 and 0.27 at Montauk and Sandy Hook respectively. The response of sea level is thus primarily wind driven. If we consider the observed spike at March 29, 60 percent of the model prediction of sea level is wind-driven at Montauk while fully 81 percent is wind driven at Sandy Hook.

Our second comparison utilizes data from Julian days 50 to 150 of 1988. The section considered is the coast of New Jersey from Sandy Hook to Lewes, Delaware, just across the mouth of Delaware Bay south of Cape May, New Jersey. We take \( y = 0 \) at Sandy Hook and predict sea level at Ventnor City, New Jersey and Lewes, Delaware approximately 130 and 200 kilometers to the south. We again take \( h_{0} = 10 \text{ m} \), \( \delta_{0} = 2 \text{ m} \) and the \( \alpha = 0.5 \times 10^{-3} \). This leads to \( L_{x} \) of 100 kilometers.
The main component of the wind is to the north, northeast at $30^\circ$ east of north. A comparison of predicted and observed sea levels is given in Figure 6. Agreement is quite reasonable given the assumptions of uniformity in the alongshore direction which is not totally satisfied in the actual case and given the lack of initial profile data. See below for a quantitative estimate of model skill.

Next, we consider a region in the South Atlantic Bight from Beaufort, North Carolina, to Charleston, South Carolina, see Figure 3. We take $y = 0$ to be Beaufort and we utilize Beaufort sea level and the along-shore wind at Wilmington, North Carolina to predict sea level at Southport, North Carolina and Charleston, South Carolina. For this shelf, we take $h = 10 \text{ m}$, $\delta = 2 \text{ m}$, and $\alpha = 0.33 \times 10^{-3}$. This leads to a value of $L$ of 150 km. Predicted (dashed) and observed (solid line) sea level at Southport and Charleston are given in Figure 6. Agreement is quite reasonable given the fact that the Gulf Stream, which is present at the shelf break, frequently is present over the shelf and must impact upon coastal sea level. We note that both predicted and observed sea level fluctuations increase in amplitude in going from Southport to Charleston.

We can quantify the skill of the model by defining a dimensionless mean squared error, $E^2$, as follows:

$$E^2 = \sum (\eta_{\text{obs}} - \eta_{\text{pred}})^2 / \sum \eta_{\text{obs}}^2$$

where the sum is over all units of observation and prediction (every six hours) over the time period of comparison. For Montauk, Sandy Hook, Ventnor, Lewes, Charleston and Southport, the calculated values of $E^2$ are 0.173, 0.122, 0.219, 0.181, 0.294, and 0.201, respectively. The model works best when fluctuations are most robust. For operational use of the model, the parameters $h/\alpha e$ and $h_0/\alpha e$ could be adjusted to minimize the error in any given region.

**CONCLUSIONS**

A simple analytical model for predicting along-shore variations in coastal sea level has been developed and applied to three US East Coast shelf regions. The three regions are from Woods Hole, Massachusetts to Sandy Hook, New Jersey, then from Ventnor, New Jersey to Lewes, Delaware and finally from Cape Hatteras, North Carolina to Charleston, South Carolina.

The good agreement between model predictions of coastal sea level and actual observations suggest that for sub-inertial frequencies, we now might have the capability of predicting the time series of sea level from Woods Hole, Massachusetts, to Charleston, South Carolina, simply by taking the time series of the alongshore component of the wind and sea level at Woods Hole; preferably this could be done in segments along which the alongshore wind is uniform or Equation 15a with alongshore varying wind could be used.

The ability to predict sea level fluctuations along the coastline of the eastern U.S. seaboard is particularly important to set inner shelf boundary
conditions for numerical shelf circulation models, to determine the non-local forcing conditions which are so important at the mouths of the large estuarine systems which are indigenous to the Mid Atlantic Bight (Wang and Elliott, 1978; Pietrafesa and Janowitz, 1988), and to help forecast the sea level response to the large wintertime atmospheric storms which buffet the entire region (Clione et al., 1993). The general success of the model suggests that it may be applicable to other regions which have a significant alongshore wind stress.

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LITERATURE CITED


APPENDIX

In this section we utilize the constant eddy viscosity model to examine the validity of some of the assumptions we have made. While this closure scheme is the simplest approach it does allow us to examine the governing equation and boundary conditions in both shallow and deep waters. Subscripts indicate partial differentiation. Following Welander (1957) we start with the following equations and boundary conditions.

\[ A_S \frac{\partial S}{\partial t} - iS = g \nabla S, \quad (A1) \]

\[ A_S \frac{\partial S}{\partial t} \bigg|_{z=0} = iT, \quad (A2) \]
\[ S = u + iv, \quad T = \gamma y / \rho_c \quad \text{and} \quad \eta_n = \eta + i \eta. \]

The solution to these equations is
\[
M = \frac{\eta_n}{f} \left( i - \cosh(\sqrt{1/2} \zeta / \delta e) \right)
\]
\[ \begin{align*}
&+ \sqrt{2i} \frac{T \delta e}{A_c} \sinh(\sqrt{1/2}(z + h) / \delta e) \\
&\div \cosh(\sqrt{1/2} \zeta) \tag{A4}
\end{align*} \]
where \( \delta e = \sqrt{A_c / 2f} \) and \( X = h / \delta e \). Integrating this result from \( z = -h \) to \( z = 0 \) yields the following complex volume flux vector.

\[ M = M_x + i M_y = \int_{-h}^{0} u \ dz + i \int_{-h}^{0} v \ dz \]
\[ = \frac{1}{f} g \eta_n \left( 1 - \frac{\tanh(\sqrt{1/2} \zeta)}{\sqrt{1/2} \zeta} \right) \]
\[ + \frac{T \delta e}{f} \left( 1 - \frac{1}{\cosh(\sqrt{1/2} \zeta)} \right). \tag{A5} \]

We note for future reference, from (A4) that
\[ A_c \frac{\partial S}{\partial z} \bigg|_{z=-h} = i A_c \frac{g \eta_n}{f} \sqrt{1/2} \tanh(\sqrt{1/2} \zeta) \]
\[ + i T / \cosh(\sqrt{1/2} \zeta). \tag{A6} \]

The vertically integrated continuity equation provides the governing equation for \( \eta \). The real and imaginary parts of equation (A5) yield \( M_x \) and \( M_y \) and the continuity equation requires that
\[ \frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0. \tag{A7} \]

This yields a linear second order partial differential equation for \( \eta \). We shall obtain the shallow water (\( X \ll 1 \)) and deep water (\( X \gg 1 \)) limits to these equations. For \( X \ll 1 \) we use the Taylor series expansion for the hyperbolic function to obtain
\[ M_x = \frac{-g \eta_n h^3 - g \eta_n h^3}{3A_c} \frac{5 T}{96 f} X^2 + \frac{T}{4f} X^4, \tag{A8a} \]
\[ M_y = \frac{-g \eta_n h^3 + g \eta_n h^3}{3A_c} \frac{15A_c}{X^2} + \frac{T}{4f} X^2. \tag{A8b} \]

From (A6), for small \( X \)
\[ A_c \frac{\partial u}{\partial z} \bigg|_{z=-h} = -g \eta_n h + 0(h^2). \tag{A9} \]

Further (A8) can be rewritten as
\[ M_x = h^3 \left( \frac{-g \eta_n}{3A_c} - g \eta_n \frac{h^3}{15A_c} \delta e \right) \frac{-5 T h}{96 f \delta e^4}. \tag{A10} \]

At the coast, \( x = 0 \), we require \( M_x(0) = 0 \). If \( h(0) \) is small but finite we obtain the boundary condition
\[ \frac{-g \eta_n - g \eta_n h(0)^2}{3A_c} \frac{-5 T h(0)}{96 f \delta e} = 0. \]

If we now let \( h(0) \) go to zero the following boundary condition holds.
\[ \frac{\partial \eta}{\partial x}(0, y) = 0. \tag{A11} \]

The governing equation for shallow water \( X \ll 1 \), utilizing (A7) and (A8) is as follows
\[ \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} + \frac{3h_x}{h} \frac{\partial \eta}{\partial x} + \frac{X^2 \partial \eta}{3 \partial y} = \frac{5 f T}{2 A_c} h. \tag{A12} \]

Both Equations A11 and A12 differ from those derived in CSANADY (1978) under his assumption that the cross-shelf pressure gradient is balanced by the Coriolis force even in shallow water. As Equation A9 shows, in shallow water the constant eddy viscosity model predicts that the cross-shelf pressure gradient is balanced by the cross-shelf bottom stress.

The deep water limit (\( X \gg 1 \)) is obtained by replacing both \( \tanh(\sqrt{1/2} \zeta) \) and \( 1 - 1 / \cosh(\sqrt{1/2} \zeta) \) by \( 1 \). Hence for \( X \gg 1 \)
\[ M_x = \frac{-g \eta_n (h - \delta e)}{f} \frac{g \delta e}{f} \eta + T / f, \tag{A13} \]
\[ M_y = \frac{-g \eta_n (h - \delta e)}{f} \frac{g \delta e}{f} \eta. \tag{A14} \]

Utilizing (A7) we obtain
\[ \delta e \left( \frac{\partial^2 \eta}{\partial x^2} + \frac{\partial^2 \eta}{\partial y^2} \right) + \frac{2h}{\delta x} \frac{\partial \eta}{\partial y} = 0 \tag{A15} \]
and for large \( X \)
\[ M_x = \frac{-g \eta_n h - g \delta e}{f} \eta + T / f. \tag{A16} \]

We have utilized the deep water Equation A15 for depths in which \( X \gg 5 \). We note that for \( X = 5 \), \( \tanh(\sqrt{1/2} \zeta) \approx 0.997 + 0.031i \) and \( 1 - 1 / \cosh \)
\sqrt{12}X = 1.13 + 0.098i. The deep water limit is thus a reasonable choice in waters at this and greater depths.

We have also specified that \( M_\tau = 0 \) at our coastal boundary, where \( X = 5 \). A full determination of the actual transport across the \( X = 5 \) isobath would require a solution of the full governing equation for shallow (\( X \ll 1 \)) intermediate (\( X = O(1) \)), and deep waters, from some small value of \( h(0) \) at the true coast to the very large values of \( h(x) \). This is beyond the scope of the present work. We note, however, some conclusions from the shallow water results for cross-shelf transport. At the coast where \( h = h(0) \), the pressure driven currents totally offset the weak wind driven cross-shelf transport. If we use the shallow water result at \( X = 1 \), the direct wind driven cross shelf transport is about 5 percent of its deep water value. The pressure gradients will partially offset even this weak transport. Thus we might expect the leakage across the \( X = 5 \) isobath to be relatively small.

We are also in a position to assess the difference between CSANADY's (1978) solution and ours. Taking his value of \( \tau \) to be \( \delta f \), the dimensionless form of his coastal boundary condition is \( \partial \eta / \partial x \) \((0, y) = T \). The solution to Equation 13a under this condition with \( \eta_c(x) = 0 \) is as follows:

\[
\eta(x, y) = -\frac{T}{\sqrt{\pi^2}} \left( \frac{2}{\sqrt{\pi}} \sqrt{y} e^{-iy} - x \text{Erfc}(y) \right)
\]

and as \( y \to \infty \), both CSANADY's solution and ours converge. Our solution shows that \( \partial \eta / \partial y \) \((0, y) \) approaches zero for large \( y \) so that for large \( y \) our coastal boundary conditions and his become the same. However, at small and intermediate values of \( y \) there is a considerable difference between the two solutions. At the coast the ratio of our solution to his is 0.40, 0.50, 0.58 and 0.67 and \( y = 0.5, 1.0, 2.0 \) and 4.0 respectively. Thus over scales on which the alongshore wind might be uniform significant differences exist.

HICKEY and POLA (1983) utilized CSANADY's (1978) to predict sea level variations along the west coast of the U.S. They considered twenty year averages of wind and sea level data. Their origin was San Diego and \( y \) increases to the north. They concluded that agreement was poor for small \( y \) and improved to the north. The CSANADY solution differs from ours in the south due to the difference in values in the wind driven term for small and intermediate values for \( y \) and in the presence of an \( \eta_c \) term which is in our solution but is not in his due to differing coastal boundary conditions. This may be the source of the poor southern predictions in the paper of HICKEY and POLA.