Hybrid Model for Bragg Scattering of Water Waves by Steep Multiply-sinusoidal Bars

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ABSTRACT


As an extension of the idea of Kirby (1986), a hybrid model (HM) is developed to study the interaction between surface waves and steep undulating bottoms consisting of slowly-varying and rapidly-varying components. The rapidly-varying components are expanded with respect to the slowly-varying components to reduce the error in the dispersion relation. The higher-order terms neglected in the extended mild-slope equation (EMSE) of Kirby (1986) under the mild-slope assumption are retained in the present HM, which improves the overall prediction capability for steep ripple beds. It is also shown that the HM is reduced to the modified mild-slope equation (MMSE) of Chamberlain and Porter (1995) when the rapidly-varying component vanishes. A robust finite element method (FEM) is developed to solve Bragg scattering over ripple beds based on HM, EMSE, and MMSE. A full linear solution is also obtained by a boundary-element-method (BEM) to be used as a bench-mark result. For doubly- and multiply-sinusoidal beds, the results of HM reasonably agreed with the BEM result and were able to predict sub-harmonic and higher-harmonic Bragg scattering. Using the developed computer programs, it is shown that the performance of Bragg breakwaters can be greatly improved by increasing the number of sinusoidal components of ripple beds. The resulting sub-harmonic and higher-harmonic resonances are shown to be significant and greatly increase the bandwidth of high performance region. An appropriate selection of the wavenumbers and phases of bottom undulation may lead to an optimal and practically viable Bragg breakwater.

ADDITIONAL INDEX WORDS: Hybrid method, Ripple beds, Mild slope equation, High order, Sub-harmonic resonance, Wave reflection, Bragg breakwater, Finite element method, Boundary element method.

INTRODUCTION

The surface wave scattering by rippled seabeds has been studied by many researchers during the past decade. When wavelengths of surface waves properly match the ripple-bed undulation, the waves reflected from successive bars become in phase and reinforce each other, which results in strong resonant reflection. This is the well-known Bragg resonance phenomenon. Davies and Heathershaw (1984) obtained linear solutions and verified this phenomenon experimentally. Mei (1985) subsequently elucidated the process close to the Bragg resonance condition. Using a boundary integral equation method, Dalrymple and Kirby (1986) have studied both resonant and nonresonant reflections by a single sinusoidal bed. Further explanation of Bragg wave scattering may also be found in Benjamin et al. (1987) and Hara and Mei (1987).

For a single sinusoidal bed, not only the primary resonance at $2k/K \sim 1$, where $k$ and $K$ are free-surface wavenumber and bottom wavenumber respectively, but also the second-harmonic resonance at $2k/K \sim 2$ can be observed (Davies et al. 1989; O'Hare and Davies, 1993). When the bottom is spatially undulated with two wave numbers $(K_1, K_2; K_2 > K_1)$, the difference- and sum-interaction resonances occur at $k = (K_2 - K_1)/2, (K_2 + K_1)/2$, respectively, which were referred to as sub-harmonic and higher-harmonic resonances by Belzons et al. (1991) and Guazzelli et al. (1992). They showed that the sub- and higher-harmonic resonances are significant although their magnitudes are smaller than those of primary peaks. To reproduce this phenomenon numerically, a step-approximation model (Guazzelli et al. 1992) or successive-application matrix (OHare and Davies, 1993) were developed, in which the bed was divided into a series of very small horizontal shelves. OHare and Davies (1993) showed that their numerical results underpredicted the measured sub-harmonic resonance.

Alternative numerical methods representing various extensions of the mild-slope equation (MSE), which was first derived by Berkhoff (1972) and Smith and Sprinks (1975) and subsequently revisited by Miles (1991), have also been developed. Using a Galerkin-eigenfunction method, Massel (1993) extended the mild-slope equation to include both evanescent modes and higher-order terms. Recently, Chamberlain and Porter (1995) proposed a modified mild slope equation (MMSE) that includes all the higher-order terms neglected in MSE. Porter and Staziker (1995) further developed the MMSE to be able to handle both evanescent modes and bottom slope discontinuity. However, when the
slope or curvature of the bottom undulation is large, the accuracy of MMSE is expected to be aggravated by the higher-order terms neglected in the dispersion relation.

On the other hand, Kirby (1986) developed an extended mild slope equation (EMSE) to predict wave propagation over slowly varying topography superposed by rapid fluctuation by expanding the rapid component with respect to the slowly-varying bottom. In this paper, he neglected higher-order terms under the mild-slope assumption $|\nabla h|kh \ll 1$, where $h$ is slowly-varying depth. Subsequently, the limitation of Kirby's model in predicting subharmonic-resonance peaks caused by steep doubly-sinusoidal bars was reported by O'Hare and Davies (1993). Since the subharmonic resonance occurs in the low frequency region, it is particularly important for the practicality of Bragg-breakwater concept. To better estimate the Bragg reflection by steep undulating bars consisting of two or more slow and rapid oscillations, the steeper components need to be expanded with respect to the slowly-varying components instead of expanding both components about the flat bed. In this manner, we can reduce the slope or curvature of rapidly-varying components and the errors related to the dispersion relation. When the slope of the slowly-varying component is not small, the higher-order terms neglected in Kirby (1986) need to be retained for the better estimate of Bragg reflection. In view of this, we propose a hybrid method (HM), where rapidly-varying components are expanded with respect to slowly-varying components as in Kirby (1986), while the higher-order terms neglected in Kirby (1986) are retained to be applicable to steep undulating bars. Through comparison with more accurate boundary-element-method (BEM)-based numerical solutions, it is shown in this paper that primary and sub and higher-harmonic Bragg resonances can be reasonably predicted by the proposed hybrid method. By retaining the higher-order terms, the amplitudes of both steep and basic components can be of the same order. A similar idea was also employed in Zhang and Melville (1992) to investigate the nonlinear interaction of long waves with short waves. They showed that better wave-kinematics estimates are possible by expanding short waves with respect to long waves instead of expanding both components about mean sea level, such as regular Stokes expansion.

Considering the problems of the Ekofisk oil field in the North Sea, Mei et al. (1988) proposed the concept of Bragg breakwaters to protect the drilling platform against prevailing waves. Similar ideas were also investigated by Kirby and Anton (1990) and Ballard et al. (1992). In their numerical examples, the effective bandwidth of the high-performance region was narrow, and thus Ballard et al. (1992) concluded that the application of a Bragg breakwater concept may be limited in practice for most US beaches. Using both primary and sub-harmonic Bragg resonances by multiple sinusoidal bottom undulations, it is shown in this paper that both magnitude and bandwidth of Bragg resonance peaks can be greatly increased, which is very important to the practicality of Bragg breakwaters.

The first part of this paper presents the hybrid-model equation based on the Green's second identity. Secondly, the paper presents comparisons with Kirby's (1986), and Chamberlain and Porter's (1995) equations. In the next section, the numerical implementation of the hybrid model based on a finite-element formulation as well as the full linear solution by an efficient BEM is explained. In the following section, the results of the hybrid model for doubly- and multiply-sinusoidal beds are compared with the corresponding experimental data and BEM solutions. Finally, a concept to enhance the performance of Bragg breakwaters is proposed.

**HYBRID-MODEL EQUATION FOR STEEP UNDULATING BARS**

We consider the propagation of a small-amplitude regular water wave over steep undulating seabed. For this, it is assumed that the flow is incompressible and irrotational and that the pressure is constant at the free surface. It is also assumed that surface-tension and nonlinear effects on the free surface can be neglected. For analysis, a fixed rectangular coordinate $(x, y, z)$ with z-axis positive upward is used and its origin is located at the calm water level. Then, the fluid motion can be described by the velocity potential $\Phi(x, y, z)$ which satisfies the following governing equation and boundary conditions:

$$
\nabla^2 \Phi + \Phi_{zz} = 0, \quad (-h' \leq z \leq 0), \quad (2.1)
$$

$$
\Phi_{zz} + g \Phi_z = 0, \quad at \ z = 0, \quad (2.2)
$$

where $g$ denotes gravitational acceleration. Since the seabottom is fixed at $z = -h'(x, y)$, the normal velocity should vanish along the bottom boundary:

$$
\Phi_z + \nabla h \cdot \nabla \Phi = 0, \quad at \ z = -h', \quad (2.3)
$$

in which $\nabla$ denotes the horizontal gradient operator i.e. $\nabla = (\partial/\partial x, \partial/\partial y)$. If the bottom function $h'(x, y)$ consists of basic (or slowly-varying) component $h(x, y)$ and steeper (or rapidly-varying) component $\delta(x, y)$, we can write

$$
h'(x, y) = h(x, y) - \delta(x, y), \quad (2.4)
$$

where $\delta$ can be regarded as a rapidly-varying oscillation component riding on a relatively slowly-varying component $h(x, y)$. Similar to Kirby's (1986) treatment, $\delta$ is expanded with respect to the basic component $h(x, y)$ when utilizing the Green's second identity. A similar idea was also used in Liu and Dangemans (1989). The equations $(2.1)$–$(2.3)$ can be combined into the following time-dependent equation (see Appendix) governing the 2-D velocity potential $\Phi(x, y)$ and the wave number $k(x, y)$:

$$
\Phi_{tt} - \nabla \cdot (C \nabla \Phi) + ( \omega^2 - k^2 \omega^2 \nu ) \Phi + g(1 - \sigma^2) \nabla \cdot (\delta \nabla \Phi) - g(2F_1 + \delta \nabla \Phi + F_2 \delta \Phi_+ + F_2 \delta \Phi) = 0 \quad (2.5)
$$

where

$$
F_1 = \sigma(1 - \sigma^2)k \nabla h + h \nabla k, \quad (2.6)
$$

$$
F_2 = \alpha_1 \nabla^2 h + \alpha_2 \nabla k \cdot \nabla h + \alpha_3 \nabla \cdot \nabla h + \alpha_4 \nabla^2 h/k + \alpha_5 \nabla^2 h/k^2 \quad (2.7)
$$

and the linear dispersion relation between circular frequency $\omega$ and wavenumber $k$:
\[ \omega^2 = gh \tanh kh. \] (2.8)

The symbols \( C(x, y) = \omega/k \) and \( C_x(x, y) = \partial \omega/\partial k \) are the wave celerity and the group velocity, respectively. The dimensionless parameters \( \alpha_i \) (i = 1, 5) are

\[ \alpha_1 = -\sigma(1 - \sigma^2)(1 - \sigma q) - 2(1 - \sigma^2)\sigma q k \delta \] (2.9)

\[ \alpha_2 = -\sigma(1 - \sigma^2)/2 + (1 - \sigma^2)\sigma q k \delta \] (2.10)

\[ \alpha_3 = q(1 - \sigma^2)(2\sigma q^2 - 5\sigma/2 - q/2) - 2(1 - \sigma^2)/2\sigma q - \sigma q k \delta \] (2.11)

\[ \alpha_4 = q(1 - \sigma^2)(1 - 2\sigma q)/4 - \sigma/4 + (1 - \sigma^2)\sigma q k \delta \] (2.12)

\[ \alpha_5 = q(1 - \sigma^2)(4\sigma q^2 - 4\sigma^2/3 - 2\sigma q - 1)/4 + \sigma/4 + (1 - \sigma^2)\sigma q^3(1 - 2\sigma q)/k \delta \] (2.13)

Throughout this paper, the notations \( q = kh \), \( Q = k(z + h) \), and \( \sigma = \tanh kh \) are used for convenience. The detailed derivation of (5) is given in Appendix. Writing the complex velocity potential in the form \( \phi = Ae^\theta \), where \( A \) and \( \theta \) are real, and considering the following relationship for monochromatic waves: \( \vec{k} = \nabla \theta \), \( \vec{a} = -\vec{e}/\vec{\theta} \), the above time-dependent wave equation is split into real and imaginary parts. The imaginary part is

\[ 2\vec{a} A + A\vec{a} + gF_z(2\nabla A + A\nabla \vec{k}) + A\nabla F_z A = 0, \] (2.14)

which leads to the wave-action conservation, if a similar treatment as in Liu and Dingemans (1989) is used. On the other hand, the real part leads to the dispersion relation, i.e., combining with the linear dispersion relation, the total dispersion relation is obtained as follows:

\[ \tilde{\omega}^2 = \omega^2 - CC_\phi \tilde{k}^2 + \frac{A_{uu}}{A} - gF_z - gF_z \nabla^2 A - gF_z \nabla A + gF_z \tilde{k}^2 \] (2.15)

where

\[ F_3 = CC_\phi g - (1 - \sigma^2)\delta \] (2.16)

\[ F_4 = \nabla(CC_\phi) g - (1 - \sigma^2)\nabla \delta + 2\delta F_1 \] (2.17)

Due to the bottom undulation, the phase function shifts from \( \theta = \kappa x - \omega t \) to a new phase \( \theta = \kappa x d x - \omega t \). The phase and group velocities, \( \vec{C} = \vec{a} \kappa \) and \( \vec{C}_g = \vec{a} \omega/\kappa \), also depend on the bottom undulation. We can see that, due to the interaction between the surface wave and undulating bottom, the dispersion relation departs from the linear dispersion. Due to this dispersion behavior, the phase of Bragg resonance will slightly shift, which can be observed in Guazzelli et al’s (1992) experiments.

The total dispersion relation (2.15) indicates that the higher-order effects depend on the bottom slope and curvature. For large bottom slopes and curvatures, nontrivial errors are expected in the leading-order solution, i.e., the approximate solution based on the linear dispersion relation. The error can be reduced by expanding the steep component with respect to the slowly-varying basic component. In this sense, the present hybrid model is different from the MMSE model proposed by Chamberlain and Porter (1995).

Correspondence to Previous Models

It is instructive to compare the present hybrid numerical model with other existing numerical models. In Kirby (1986), \( h(x, y) \) is to satisfy the mild-slope assumption, i.e., \( |\nabla h|k \approx 1 \). Therefore, all the terms containing \( F_1 \) and \( F_2 \) in Eq.(5) were neglected. Then, Kirby’s equation can be recovered as follows:

\[ \phi_{tt} - \nabla \cdot (CC_\phi \nabla \phi) + (\omega^2 - k^2 CC_\phi \tilde{h} + g(1 - \sigma^2)\nabla \cdot (\delta \nabla \phi) = 0 \] (3.1)

For monochromatic waves over constant mean depth, the above equation is reduced to

\[ CC_\phi \tilde{v}^2 + k^2 CC_\phi \tilde{h} - g(1 - \sigma^2)\nabla \cdot (\delta \nabla \phi) = 0 \] (3.2)

Although the idea of superposing the rapid component on a mild-slope bottom is embodied in Eq.(3.1), Kirby (1986, 1993) actually used (3.2) for the calculations for ripple beds. Whereas, Tsay et al. (1989) used Eq.(3.1) to study wave transformation over the slowly-varying topography superposed by rapidly-varying components. When the above equations are used for steep undulating bars, the error related to the higher-order terms is expected to be large, which will be shown in the forthcoming numerical examples.

If \( \delta = 0 \) and \( h = h' \), the modified mild-slope equation (MMSE) can be obtained from Eq.(2.5):

\[ \phi_{tt} - \nabla \cdot (CC_\phi \nabla \phi) + (\omega^2 - k^2 CC_\phi \tilde{h} - gF_z \tilde{h} = 0 \] (3.3)

where

\[ F'_z = \alpha'_i(\tilde{\nabla} \tilde{h})^2 k + \alpha'_i(\tilde{\nabla} \tilde{h}) + \alpha'_i(\tilde{\nabla}^2 \tilde{h} k + \alpha'_i(\tilde{\nabla}^2 \tilde{h} k^2 = \alpha'_i(\tilde{\nabla} \tilde{h})^2 k^2 \] (3.4)

and the dimensionless parameters, \( \alpha'_i \) (i = 1, \ldots, 5), are

\[ \alpha'_1 = -\sigma(1 - \sigma^2)(1 - \sigma q) \] (3.5)

\[ \alpha'_2 = -\sigma(1 - \sigma^2)/\sigma q \] (3.6)

\[ \alpha'_3 = q(1 - \sigma^2)(2\sigma - 5\sigma/2 - 2)/2 \] (3.7)

\[ \alpha'_4 = q(1 - \sigma^2)(1 - 2\sigma q)/4 - \sigma/4 \] (3.8)

\[ \alpha'_5 = q(1 - \sigma^2)(4\sigma^2 - 4\sigma^2/3 - 2\sigma q - 1)/4 + \sigma/4. \] (3.9)

In this case, the total dispersion relation is given by

\[ \tilde{\omega}^2 = \omega^2 + CC_\phi(\tilde{k}^2 - k^2) + \frac{A_{uu}}{A} - CC_\phi \frac{\nabla^2 A}{A} - \frac{\nabla(CC_\phi) \nabla A}{A} - gF_z \] (3.10)

If the frequency is fixed and the temporal variation of wave amplitude is not considered, the effective wave number becomes

\[ \tilde{k}^2 = k^2 + \frac{\nabla^2 A}{CC_\phi A} + \frac{\nabla(CC_\phi) \nabla A}{CC_\phi A} + \frac{gF_z}{CC_\phi A} \] (3.11)

Discarding the last term, the effective wave number of the MSE (Liu, 1990, p35) can be recovered.

From the principle of phase conservation (Mei, 1983), we can write \( \vec{k} + \vec{v} = 0 \). For steady state, \( \vec{k} = 0 \) and \( \vec{v} = 0 \),
which implies that the wavenumber must change accordingly with the bottom variation. If the linear dispersion equation (2.8) is used, leading-order solutions can be obtained, as in Massey (1993) and Chamberlain and Porter (1995), with the following relations:

\[
\frac{\nabla^2 k}{k} = \frac{\beta_1 \nabla h}{h}, \tag{3.12}
\]

and

\[
\frac{\nabla^2 k}{k} = \beta_1 \frac{\nabla h}{h} + \beta_2 \frac{(\nabla h)^2}{h^2}. \tag{3.13}
\]

where

\[
\beta_1 = -q(1 - \sigma^2)\gamma \tag{3.14}
\]

\[
\beta_2 = 2q^2(1 - \sigma^2)\gamma - \alpha_1 \gamma \tag{3.15}
\]

\[
\gamma = \sigma + q(1 - \sigma^2). \tag{3.16}
\]

Using the notations of Massey (1993) and Chamberlain and Porter (1995), the following relations can be found:

\[
R_{10} = 2q_1 \alpha_1' + \alpha_1' \beta_1 + \alpha_1' \beta_2 + \alpha_1' \beta_3 - \alpha_1' \beta_4/q, \tag{3.17}
\]

\[
R_{20} = 2q_2 \alpha_2' + \alpha_3' \beta_1 + \alpha_4' \beta_1 - \alpha_2' \beta_4, \tag{3.18}
\]

\[
u_1' = \alpha_1' + \alpha_1' \beta_1, \tag{3.19}
\]

\[
u_2' = k(\alpha_1' + \alpha_1' \beta_1 + \alpha_2' \beta_2 + \alpha_3' \beta_3 + \alpha_4' \beta_4/q^2). \tag{3.20}
\]

Therefore, both Eq. (34) of Massey (1993) and Eq. (2.12) of Chamberlain and Porter (1995) are recovered. The numerical values of \(R_{10}\) and \(R_{20}\) have been checked against those of Massey (1993).

**Numerical Implementation**

**Finite Element Method (FEM) Based On a Weak Form**

For steady-state 2-D waves, Eq.(2.5) becomes

\[
\phi_{w0} + p_1 \phi_x + p_2 \phi_y = 0 \tag{4.1}
\]

where \(p_1 = (CC_1)/F_2 - g(1 - \sigma^2)\delta/F_3 + 2gF_1\delta/F_2\) and \(p_2 = kCC_2/F_3 + g(F_1\delta + F_2\delta)/F_3\). The boundary conditions (e.g. Kirby, 1986) for the truncation surfaces of a rippled bed of length \(L\) are

\[
\phi_x = -ik(\phi - 2\beta_1) \quad (x_1 \leq 0) \tag{4.2}
\]

\[
\phi_x = i\kappa \phi \quad (x_2 \geq L) \tag{4.3}
\]

where \(\phi_x = e^{ikx}\) is the incident wave of unit amplitude, \(x_1\) and \(x_2\) represent the upwave and downwave limits of the computational grid. In Chandrasekera and Cheung (1997), local waves are also included in the outer solution. Our numerical calculations have shown that the matching conditions of Porter and Staziker (1995) have virtually no effect in the case of ripple-bed problems.

Multiplying the left-hand side of (4.1) by a weight function \(w\) and integrating over the domain \((0,L)\), we obtain the following weighted-residual equation:

\[
\int_0^L (\phi_{w0} + p_1 \phi_x + p_2 \phi_y) \, w \, dx = 0 \tag{4.4}
\]

Mathematically, (4.4) states that the numerical error in Eq.(4.1) needs to be zero in the weighted-integral sense. From integration by parts, a weak form of equation (4.4) can be obtained:

\[
\int_0^L (\phi_{w0} - p_1 \phi_x - p_2 \phi_y) \, dx - \left[ w \phi_x \right]_0^L = 0 \tag{4.5}
\]

The trading of differentiability from \(\phi\) to \(w\) can only be performed if it leads to boundary terms that are physically meaningful. If we choose \(w = \phi\), the resulting boundary term \((\phi)\) has the physical meaning of energy flux through a section after multiplying by \(CC_3\). It is easy to find that the primary variable and the secondary variable are \(\phi\) and \(\phi_x\), respectively. Thus, \(\phi \phi_x\) is the natural boundary condition (NBC). Using the notations of Reddy (1993), we write

\[
B(\phi, \phi) = \int_0^L (\phi_{w0} - p_1 \phi_x - p_2 \phi_y) \, dx - \left[ i k \phi w \right]_0^L
\]

and

\[
l(w) = -\left[ 2ikw\phi_x \right]_{-\infty}^\infty \tag{4.6}
\]

are bilinear and linear forms, respectively.

Most finite element models for water wave problems are established based on the functional formulation, for example, the hybrid element method of Mei (1983) is based on the variation of a functional. Here, only the weak form of the new equation and the NBC are needed. Thus, any approximations for the primary variable \(\phi\) are not necessary outside the computational domain. Also, as noted in Reddy (1993), not all differential equations admit the functional formulation, and in order for the functional to exist, the bilinear form must be symmetric in its arguments, as opposed to this case. Substituting the assumed approximate solution into the above weak form and following the procedure of Reddy (1993), a robust FEM computer program is developed. Considering the fact that the coefficients \(p_1\) and \(p_2\) contain second-order spatial derivatives, cubic shape functions are used here. A convergence test was carried out and 120 elements (or 361 nodal points) give sufficient accuracy for all the cases and frequency range considered here.

**Boundary Element Method (BEM)**

To obtain full linear solutions, a boundary integral equation method based on simple sources is developed. The sources and dipoles are distributed over the entire boundary to be used for arbitrary-topography problems. A Sommerfeld radiation (outgoing-wave) condition was applied at two vertical truncation boundaries. The truncation boundaries are located \(4h\) away from the end points of ripple beds to ensure that exponentially decaying local-wave (or evanescent-wave) effects can be neglected. The asymptotic behavior of local waves with radial distance is discussed in detail in Kim (1991). Applying Green’s theorem and imposing requisite boundary conditions, we obtain
having only one sinusoidal component, the present hybrid model degenerates into either EMSE (KIRBY, 1986) if $h = constant$ or MMSE (CHAMBERLAIN and PORTER, 1995) if $\delta = 0$. The numerical results of MSE, EMSE, and MMSE for this particular case are presented in Figure 1 with the experimental data of DAVIES and HEATHERSHAW (1984). We can see that the numerical results of MMSE and EMSE are in good agreement with experimental data, while the MSE result significantly underestimates the primary peak. This example shows that the original mild slope assumption $\nabla h/kh \ll 1$ is very restrictive and not suitable for ripple-bed scattering problems (KIRBY, 1986). For comparison, the result for $b/h_o = 0.32$ is also presented in Figure 1. It is seen that both the amplitude and bandwidth are increased compared to the case of $b/h_o = 0.16$.

Next, doubly-sinusoidal bars consisting of both slowly-varying and rapidly-varying components are considered. The bed form can be expressed by

$$h'(x) = h_o$$

$$h'(x) = h_o - b[\sin(pKx) + \sin(mKx)]$$

$$h'(x) = h_o$$

where $h_o$ is mean water depth, $p$ and $m$ are constants, and $m/p$ is the ratio of the longer- to shorter-bed wavelengths. Let us first consider an ideal example for HM, where wavenumbers of two bottom undulations are quite different, i.e., $h'(x) = h_o - b[\sin(0.3Kx) + 1/2 \sin(1.5Kx)]$ with $n = 3$ and $b/h_o = 0.33$. In Figure 2, the results of EMSE, MMSE, and HM are compared with computationally more expensive BEM results which can be regarded as exact solutions. It seems that both HM and MMSE can reasonably predict the wave reflection by doubly sinusoidal bars but the EMSE appears to be reliable only for the primary peak of the slowly-varying component.

In Figures 3-4, the cases of $n = 8$, $p = 1$, $m = 1.5$, and $b/
$h_0 = 0.25$ and $0.4$ are considered to compare with existing experimental data. This case was also studied by O'HARE and DAVIES (1993) and GUAZZELLI et al. (1992). Numerical results of the hybrid model as well as MMSE and EMSE for these cases are presented and compared with GUAZZELLI et al.'s (1992) experimental data. In Figure 4, they are also compared with the BEM solution which is believed to be exact in the context of linear potential wave theory. For the hybrid model, $h(x) = h_0 - b \sin(Kx)$ and $b(x) = b \sin(1.5Kx)$ are used. Although this is not an ideal example for HM, the hybrid model still gives reasonable results due to the reduction of error in the dispersion relation. The strong interaction of surface waves with the doubly-periodic bottom at multiple locations, i.e., not only the primary Bragg reflection at $2k/K = 1$ but also subharmonic resonance at $2k/K = 0.5$ and higher-harmonic resonance at $2k/K = 1.5$, can be observed in the two cases. When compared with the BEM result, the present hybrid model reasonably predicts the magnitudes of Bragg resonance peaks. O'HARE and DAVIES (1993) pointed out that the EMSE underestimated the subharmonic resonance at $2k/K = 0.5$ which can also be observed in these examples. The error results from the use of constant wavenumber $(o^2 = gh \tanh k h_0)$ due to the mild-slope assumption for these kinds of relatively steep ripple-bed problems. It is also seen that the relative importance of higher-harmonic resonance increases with $b/h_0$. The numerical prediction tends to deviate farther from measured values as the bottom slope increases. This can partly be attributed to the increased energy loss due to viscous effects. Another notable feature in these figures is the shift of experimental peaks or those of BEM solutions to lower wavenumber ratio (or lower wave frequency) when compared with the results of the approximation methods (EMSE, MMSE, and HM) based on propagation modes only. By including several evanescent modes, GUAZZELLI et al. (1992) also showed that the frequency shift is partly due to the effects of evanescent waves. The BEM solution including all the evanescent modes supports GUAZZELLI et al.'s speculation. When compared with Figure 1, we can see that the performance of a doubly-sinusoidal Bragg breakwater can be greatly enhanced by utilizing additional sub- and higher-harmonic resonances.

In Figure 5, we compared the performance of the same doubly-periodic bottom of two different lengths i.e. $n = 4$ and $n = 8$. The HM was used for this computation. We can see that the overall effectiveness is reduced by about 20% when the length is shortened by half. Interestingly, many narrow sub-peaks disappear when the length is shortened. Comparing with Figures 3–4, it seems better to increase the amplitude of bottom undulation than to increase the length of the ripple bed to enhance the overall wave-blocking efficiency for a given amount of construction material. However, in practice, $b/h_0$ cannot not be too large. Otherwise, the bars would begin to look more like submerged breakwaters than Bragg breakwaters. In the next figure, we investigated the sensitivity of wave-blocking performance to the change of phases of two sinusoidal components; 

$$h'(x) = h_0 - b \sin(Kx + \psi) + \sin...$$
Figure 6. Doubly-sinusoidal bed with phase shift.

(1.5Kx - ψ) and n = 8. As can be seen in Figure 6, the Bragg resonance depends primarily on the bottom wavenumbers and is almost not influenced by the phase of individual sinusoid. In view of the observation made in the above, an effective composite Bragg breakwater can be proposed, for which half of the bottom profile considered in Figure 4 is replaced by a new ripple bed consisting of different sinusoids i.e. \[ h'(x) = h_0 - b[\sin(Kx) + \sin(1.75Kx)] \]. Then, we expect not only an additional primary peak at \( 2k/K = 1.75 \) but also additional subharmonic Bragg resonances at \( 2k/K = 0.75, 0.25 \) and additional higher-harmonic resonances at \( 2k/K = 1.75, 3.25 \) etc. In this manner, we can greatly increase the bandwidth of high performance region. This concept can numerically be verified, as shown in Figure 7, where the hybrid model result correlates well with the BEM solution. The slight shift of peaks of MMSE, EMSE, and HM, which was already discussed in this section, can again be observed in this example. Due to the narrow bandwidth of high performance, BAILARD et al. (1992), for example, maintained that the Bragg breakwater concept may not be practical for most US beaches. However, this example typically shows that the overall effectiveness of Bragg breakwaters can be greatly enhanced if they are properly designed.

In contrast to the composite sinusoidal bed considered in the above, multiple sinusoidal components can simply be superposed to improve the overall wave-blocking performance. Figure 8 shows an example of a bar consisting of three different sinusoidal components i.e. \[ h'(x) = h_0 - b[\sin(0.75Kx) + \sin(Kx) + \sin(1.5Kx)] \]. For HM, \( h(x) = h_0 - b[\sin(0.75Kx) + \sin(Kx)] \) and \( \delta(x) = b\sin(1.5Kx) \) are used. It can be seen from Fig. 8 that the HM again reasonably predicts the sub- and higher-harmonic Bragg resonances by multiply sinusoidal bars. The bandwidth of high performance region is again greatly increased due to many possible sub- and higher-harmonic resonances. In Figure 9, the performance of the same beds with three sinusoidal components for three different \( b/h_0 \) values is compared. As expected, the wave blocking efficiency is monotonically decreased as bar amplitudes become smaller. The decay rate is larger for greater \( k/K \) values.

Finally, another ripple bed consisting of four different si-
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\[ h'(x) = h_0 - b \left[ \sin(0.75Kx + \psi) + \sin(Kx + \psi) + 0.5 \sin(1.5Kx - \psi) + 0.5 \sin(1.75Kx - \psi) \right] \]

is considered. For HM, \( h(x) = h_0 - b[\sin(0.75Kx) + \sin(Kx)] \) and \( \delta(x) = 0.5b[\sin(1.5Kx) + \sin(1.75Kx)] \) are used. When more than two sinusoids exist, the choice of slow and fast components can be different. In Figure 10, HM for \( \psi = 0 \) is again in reasonable agreement with BEM. In Figure 11, the effect of phase shift among four sinusoidal components is shown. It is seen that its influence is again not significant, as was observed in Figure 6.

**CONCLUDING REMARKS**

As an extension of the idea of Kirby (1986), a hybrid model (HM) is developed to study the interaction between surface waves and steep undulating bottoms which can be represented as slowly-varying and rapidly-varying components. The rapidly-varying components are expanded with respect to the slowly-varying components to reduce the higher-order slope and curvature terms neglected in the dispersion relation. On the other hand, the higher-order terms, which were neglected in the extended mild-slope equation (EMSE) of Kirby (1986) under the mild-slope assumption, are retained in the present HM, which enhances the overall prediction capability for steep ripple beds. It is also shown that HM is reduced to the modified mild-slope equation (MMSE) of Chamberlain and Porter (1995) when the rapidly-varying components are removed. A robust finite element method (FEM) is developed to solve Bragg scattering over ripple beds based on HM, EMSE, and MMSE. A full linear solution is also obtained by a computationally more expensive boundary-element-method (BEM) to be used as a benchmark result. The present method can be extended to oblique incident waves, and then the superposition principle can be applied for directionally random waves.

For doubly- and multiply-sinusoidal beds, the results of HM are shown to better correlate with the BEM result than EMSE. The advantage of HM over MMSE was not clearly shown because all the computed results were for practically horizontal bottoms of relatively short distances. For very steep bars, the discrepancy between numerical prediction and measurement is not small, and it can be attributed to increased viscous effects for larger peaks. It is also observed that the peaks of BEM solutions tend to shift toward lower wave frequencies compared to the propagation-mode-only approximation methods, such as HM, EMSE, and MMSE. The phase-shift phenomenon can be attributed to the effects of evanescent modes as well as the error caused by the use of the linear dispersion relation.

Using the developed computer programs, the performance of singly-, doubly-, and multiply-sinusoidal beds is compared. A composite bed consisting of two different combinations of double sinusoids was also assessed. It is shown that the performance of Bragg breakwaters can be greatly improved by
increasing the number of sinusoidal components of ripple beds. The performance is not very sensitive to the phase difference of individual sinusoidal components and high efficiency can be achieved without using very long ripple beds. The wave-blocking effectiveness is in general monotonically increased as bottom amplitude to mean waterdepth ratio increases. When multiple sinusoids are used, in addition to additional primary peaks, the resulting sub-harmonic and higher-harmonic resonances are also found to be significant and greatly increase the bandwidth of the high performance region. Therefore, an appropriate selection of the wavenumbers and phases of bottom undulation may lead to the optimal and practically viable Bragg breakwaters for various sea conditions.

**APPENDIX: DERIVATION OF THE TIME-DEPENDENT WAVE EQUATION**

The depth-integrated wave equation for monochromatic linear waves propagating over ripple beds can be formulated following the Green's-second-identity method of SMITH and SPRINKS (1975) and KIRBY (1986). In this paper, all the higher-order terms neglected in Kirby (1986) are retained such that \(|\partial h/\partial h|\) may not be small. The solution may be expressed as

\[ \Phi(x, y, z, t) = f(q, Q)\phi(x, y, t) + \text{(nonpropagating modes)} \]  

where

\[ f = \cosh Q/\cosh q \]  

is a function of \(z, k, h\). After neglecting nonpropagating (or evanescent) modes and applying the Green's second identity to \(f\) and \(\Phi\), we obtain

\[ \int_{-h}^{0} (f\Phi_{zz} - \Phi f_{zz}) \, dz = \left[f\Phi_{z} - \Phi f_{z}\right]_{0}^{-h} \]

or

\[ \int_{-h}^{0} (f\nabla^{2}\Phi + \Phi f_{zz}) \, dz = \left[-f\Phi_{z} + \Phi f_{z}\right]_{0}^{-h} \]  

Considering

\[ f_{zz} = k^{2}f \]

\[ \nabla^{2}\Phi = f_{\nabla^{2}f} + \phi\nabla f \]

\[ \nabla^{2}f = f_{\nabla^{2}f} + 2f_{\nabla^{2}f} + \phi\nabla^{2}f \]

\[ \Phi_{|_{z=-h}} = (\delta h - \nabla h) \cdot (f\nabla^{2}f + \phi\nabla f) + \delta\nabla^{2}\Phi \]  

and inserting (A4) into (A3), the following equation can be obtained:

\[ \int_{-h}^{0} (f\Phi_{zz} + \Phi f_{zz}) \, dz = \left[f\Phi_{z} - \Phi f_{z}\right]_{0}^{-h} \]

\[ = \left[f\Phi_{z} - \Phi f_{z}\right]_{0}^{-h} + \int_{-h}^{0} (\Phi_{zz} + \omega^{2}\Phi) \, dz = \int_{-h}^{0} (\Phi_{zz} + \omega^{2}\Phi) \, dz \]  

In (A4), it can be proved that all the other terms (e.g. \(\delta h - \nabla h) \cdot (f\nabla^{2}f)\), \(1/\delta h\Phi_{zz} \), etc.) neglected in the Taylor expansion of the bottom condition are at least one-order higher than \(\delta \nabla^{2}\Phi\). Every term in (A5) can be evaluated using the following relationship

\[ \nabla f = f_{z}\nabla h + f_{h}\nabla k \]

\[ \nabla^{2}f = f_{h\nabla h} + f_{h\nabla k} + f_{h\nabla k} + f_{h\nabla k} + f_{h\nabla k} + f_{h\nabla k} \]

where \(f_{z} = \delta f/\partial h, f_{h} = \delta f/\partial k, f_{h} = \delta f/\partial k, f_{h} = \delta f/\partial k, \) and derived in the following:

\[ f_{h} = (2\sigma_{2}q - \sigma_{2}q)\cosh Q/\cosh q \]

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Applying Leibniz's rule, the following identity can be obtained:

\[ \int_{-h}^{0} (\nabla f_{zz} + 2f_{\nabla^{2}f} \cdot \nabla f) \, dz + f_{2}\nabla h \cdot \nabla f \]  

\[ = \nabla \cdot (CC_{f}\nabla f) \]  

Using the following integrations

\[ g \int_{-h}^{0} k^{2}f_{zz} \, dz = k^{2}CC_{f} \phi \]

\[ 2\kappa \int_{-h}^{0} \sinh Q \cosh Q \, dz = \sigma^{2}/(1 - \sigma^{2}) \]

\[ 4\kappa \int_{-h}^{0} \sinh Q \cosh Q \, dz = (q(1 + \sigma^{2}) - \sigma)(1 - \sigma^{2}) \]

\[ 4\kappa \int_{-h}^{0} \sinh Q \cosh Q \, dz = (q^{2}(-1 + \sigma^{2}) + 2q_{2} - \sigma^{2})(1 - \sigma^{2}) \]

\[ 4\kappa \int_{-h}^{0} \sinh Q \cosh Q \, dz = (q^{2}(1 + \sigma^{2}) + 2q_{2} - \sigma^{2})(1 - \sigma^{2}) \]

and substituting (A6) and (A7) into (A5), the time-dependent equation (2.5) is finally obtained.

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